

Plasma Relaxation and Topological Aspects in Hall Magnetohydrodynamics

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Abstract

Parker's formulation of isotopological plasma relaxation process in magnetohydrodynamics (MHD) is extended to Hall MHD. The torsion coefficient α in the Hall MHD Beltrami condition turns out now to be proportional to the "*potential vorticity*." The Hall MHD Beltrami condition becomes equivalent to the "*potential vorticity*" conservation equation in two-dimensional (2D) hydrodynamics if the Hall MHD Lagrange multiplier β is taken to be proportional to the "*potential vorticity*" as well. The winding pattern of the magnetic field lines in Hall MHD then appears to evolve in the same way as "*potential vorticity*" lines in 2D hydrodynamics.

1. Introduction

A significant class of exact solutions of the equations governing magnetohydrodynamics (MHD) emerges under the Beltrami condition - the local current density is proportional to the magnetic field - the *force-free* state (Lundquist [1], Lust and Schluter [2]). These Beltrami solutions turned out to correlate well with real plasma behavior (Priest and Forbes [3], Schindler [4]). Parker [5] - [7] showed that, in certain plasma relaxation processes, the Beltrami condition is indeed equivalent to the vorticity conservation equation in two-dimensional (2D) hydrodynamics (and the Lagrange multiplier α turned out to be proportional to vorticity).

In a high- β plasma, on length scales in the range $d_e < \ell < d_i$, where d_s is the skin depth, $d_s \equiv c/\omega_{ps}$, $s = i, e$ (i and e referring to the ions and electrons, respectively), the electrons decouple from the ions. This results in an additional transport mechanism for the magnetic field via the Hall current (Sonnerup [8]), which is the ion-inertia contribution in Ohm's law. The Hall effect leads to the generation of whistler waves whose,

- frequency lies between ion-cyclotron and electron-cyclotron frequencies ω_{ci} and ω_{ce} , respectively,
- phase velocity exceeds that of Alfvén waves for wavelengths parallel to the applied magnetic fields less than d_i .

Further, the decoupling of ions and electrons in a narrow region around the magnetic neutral point (where the ions become unmagnetized while the electrons remain magnetized) allows for rapid electron flows in the ion-dissipation region and hence a faster magnetic reconnection process in the Hall MHD regime (Mandt et al. [9]).

The purpose of this paper is to extend Parker's [5] - [7] considerations to Hall MHD and investigate the evolution of the winding pattern of the magnetic field lines in Hall MHD.

2. Beltrami States in Hall MHD

The Hall MHD equations (which were formulated by Lighthill [10] following his far-sighted recognition of the importance of the Hall term in the generalized Ohm's law) are (in usual notations),

$$\frac{\partial \mathbf{\Omega}_i}{\partial t} = \nabla \times (\mathbf{v}_i \times \mathbf{\Omega}_i) \quad (1)$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{c} \mathbf{v}_i \times \mathbf{B} - \frac{1}{nec} \mathbf{J} \times \mathbf{B} \quad (2)$$

where n is the number density of ions (or electrons) and $\mathbf{\Omega}_i$ is the generalized vorticity,

$$\mathbf{\Omega}_i \equiv \boldsymbol{\omega}_i + \boldsymbol{\omega}_{ci}, \quad \boldsymbol{\omega}_i \equiv \nabla \times \mathbf{v}_i, \quad \boldsymbol{\omega}_{ci} \equiv \frac{e\mathbf{B}}{m_i c}. \quad (3)$$

Here, we have considered an incompressible, two-fluid, quasi-neutral plasma and have neglected the electron inertia.

Equations (1) and (2) have the Hamiltonian formulation (Shivamoggi [11]),

$$H = \frac{1}{2} \int_V \left[\boldsymbol{\psi}_i \cdot \boldsymbol{\Omega}_i + \frac{1}{c} \mathbf{A} \cdot (\mathbf{J} - ne\mathbf{v}_i) \right] dV \quad (4)$$

where,

$$m_i n \mathbf{v}_i \equiv \nabla \times \boldsymbol{\psi}_i \quad (5)$$

and V is the volume occupied by the plasma.¹ Further, we have put $|\boldsymbol{\psi}_i| = 0$ on the boundary ∂V , and have rendered $\boldsymbol{\psi}_i$ unique by imposing the gauge condition

$$\nabla \cdot \boldsymbol{\psi}_i = 0. \quad (6)$$

We choose $(\boldsymbol{\Omega}_i, \mathbf{A})$ to be the canonical variables, and take

$$J \equiv \begin{pmatrix} -\nabla \times \left(\frac{\boldsymbol{\Omega}_i}{m_i n} \times (\nabla \times (\cdot)) \right) & \mathbf{0} \\ \mathbf{0} & \frac{c\mathbf{B}}{ne} \times (\cdot) \end{pmatrix} \quad (7)$$

as a $(\boldsymbol{\Omega}_i, \mathbf{A})$ -dependent differential operator which produces a skew-symmetric transformation of vector functions vanishing on ∂V and satisfies a closure condition on an associated symplectic two-form (Olver [12]).

The Hamilton equations are then

$$\begin{pmatrix} \frac{\partial \boldsymbol{\Omega}_i}{\partial t} \\ \frac{\partial \mathbf{A}}{\partial t} \end{pmatrix} = J \begin{pmatrix} \frac{\delta H}{\delta \boldsymbol{\Omega}_i} \\ \frac{\delta H}{\delta \mathbf{A}} \end{pmatrix} \quad (8)$$

which are just equations (1) and (2). Here, $\delta H / \delta \mathbf{q}$ is the variational derivative.

The Casimir invariants for Hall MHD are solutions of the equations,

$$J \begin{pmatrix} \frac{\delta \mathcal{C}}{\delta \boldsymbol{\Omega}_i} \\ \frac{\delta \mathcal{C}}{\delta \mathbf{A}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \quad (9)$$

It may be verified that two such solutions are

$$\begin{pmatrix} \frac{\delta \mathcal{C}_{(1)}}{\delta \boldsymbol{\Omega}_i} \\ \frac{\delta \mathcal{C}_{(1)}}{\delta \mathbf{A}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{B} \end{pmatrix} \quad (10)$$

¹(5) implies

$$\frac{\partial n}{\partial t} = 0$$

in accord with the assumption that the plasma is incompressible.

or

$$\mathcal{C}_{(1)} = \int_V \mathbf{A} \cdot \mathbf{B} \, dV \quad (11)$$

as with classical MHD, and

$$\begin{pmatrix} \frac{\delta \mathcal{C}_{(2)}}{\delta \boldsymbol{\Omega}_i} \\ \frac{\delta \mathcal{C}_{(2)}}{\delta \mathbf{A}} \end{pmatrix} = \begin{pmatrix} \frac{e \mathbf{A}}{m_i c} + \mathbf{v}_i \\ \left(\frac{e}{m_i c} \right)^2 \mathbf{B} \end{pmatrix} \quad (12)$$

or

$$\mathcal{C}_{(2)} = \int_V \left(\frac{e \mathbf{A}}{m_i c} + \mathbf{v}_i \right) \cdot \boldsymbol{\Omega}_i \, dV. \quad (13)$$

$\mathcal{C}_{(1)}$ is the total magnetic helicity and $\mathcal{C}_{(2)}$ is the total generalized ion cross helicity.

A significant class of exact solutions of the Hall MHD equations (1) and (2) emerges as the end result of the isotopological energy-lowering Beltramization process. Thus, minimization of H , keeping $\mathcal{C}_{(1)}$ fixed, gives

$$\frac{\delta H}{\delta \mathbf{A}} = \lambda_{(1)} \frac{\delta \mathcal{C}_{(1)}}{\delta \mathbf{A}} \quad (14)$$

or

$$\frac{1}{c} (\mathbf{J} - ne \mathbf{v}_i) = \lambda_{(1)} \mathbf{B} \quad (15)$$

which is the pseudo-force-free state.

On the other hand, minimization of H , keeping $\mathcal{C}_{(2)}$ fixed, gives

$$\frac{\delta H}{\delta \boldsymbol{\Omega}_i} = \lambda_{(2)} \frac{\delta \mathcal{C}_{(2)}}{\delta \boldsymbol{\Omega}_i} \quad (16)$$

or

$$m_i n \mathbf{v}_i = \lambda_{(2)} \boldsymbol{\Omega}_i \quad (17)$$

which is the generalized Alfvénic state.

Combining (15) and (17), we obtain for the Hall MHD Betrami state (Turner [13]),

$$\frac{m_i}{e} \nabla \times \mathbf{B} - \left(\lambda_{(1)} \frac{m_i}{e} + \frac{e}{m_i c} \lambda_{(2)} \right) \mathbf{B} = \lambda_{(2)} \boldsymbol{\omega}_i. \quad (18)$$

3. Plasma Relaxation in an Applied Uniform Magnetic Field

Consider now, following Parker [5] - [7], a plasma in an applied uniform magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{i}}_z$ and confined between two infinite parallel planes $z = 0$ and L , which relaxes²

²In this process, the magnetic field lines extending between the planes $z = 0$ and L are wrapped around and intermixed by the motion of their foot points on these planes (Parker [5] - [7]).

isotopologically toward the lowest available energy state described by equation (18) written in the form

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} + \beta \boldsymbol{\omega}_i. \quad (19)$$

The MHD Lagrange multiplier α may be interpreted as the torsion coefficient while β is the Hall MHD Lagrange multiplier.

Suppose this process exhibits slow variations in the z-direction, characterized by the slow spatial scale,

$$\xi \equiv \epsilon z, \quad \epsilon \ll 1. \quad (20)$$

Let the magnetic field involved in this process be given by

$$\mathbf{B} = \langle \epsilon B_0 b_x, \epsilon B_0 b_y, B_0 (1 + \epsilon b_z) \rangle \quad (21)$$

and the Lagrange multipliers α and β be given by

$$\alpha = \epsilon a, \quad \beta = \epsilon b. \quad (22)$$

Using (20) - (22), equation (17) may be written as

$$v_x = \sigma (c_1 \epsilon b_x + \omega_x) \quad (23a)$$

$$v_y = \sigma (c_1 \epsilon b_y + \omega_y) \quad (23b)$$

$$v_z = \sigma [c_1 (1 + \epsilon b_z) + \epsilon \omega_z]. \quad (23c)$$

The out-of-plane (or *toroidal*) ion flow ($v_z \neq 0$) is peculiar to Hall MHD. Here, σ and c_1 are appropriate constants. Equation (19) leads to

$$\frac{\partial b_z}{\partial y} - \epsilon \frac{\partial b_y}{\partial \xi} = \epsilon a b_x + \epsilon b \omega_x \quad (24a)$$

$$\epsilon \frac{\partial b_x}{\partial \xi} - \frac{\partial b_z}{\partial x} = \epsilon a b_y + \epsilon b \omega_y \quad (24b)$$

$$\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} = a (1 + \epsilon b_z) + \epsilon b \omega_z \quad (24c)$$

and the divergence-free condition on \mathbf{B} leads to

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \epsilon \frac{\partial b_z}{\partial \xi} = 0. \quad (25)$$

On the other hand, taking the divergence of equation (19), we obtain

$$\mathbf{B} \cdot \nabla \alpha + \boldsymbol{\omega}_i \cdot \nabla \beta = 0 \quad (26)$$

which, on using (20) - (22), leads to

$$b_x \frac{\partial a}{\partial x} + b_y \frac{\partial a}{\partial y} + (1 + \epsilon b_z) \frac{\partial a}{\partial \xi} + \omega_x \frac{\partial b}{\partial x} + \omega_y \frac{\partial b}{\partial y} + \epsilon \omega_z \frac{\partial b}{\partial \xi} = 0. \quad (27)$$

Equations (24a) and (24b) imply,

$$b_z \sim O(\epsilon). \quad (28)$$

Using (28), equation (25) leads to, to $O(1)$,

$$b_x = \frac{\partial \psi}{\partial y}, \quad b_y = -\frac{\partial \psi}{\partial x} \quad (29)$$

for some magnetic flux function $\psi = \psi(x, y)$.

Using (29), we obtain from (23), to $O(\epsilon)$,

$$\frac{\partial v_x}{\partial y} = \sigma \left(c_1 \epsilon \frac{\partial^2 \psi}{\partial y^2} + \epsilon \frac{\partial^2 v_z}{\partial y^2} \right) \quad (30a)$$

$$\frac{\partial v_y}{\partial x} = \sigma \left(-c_1 \epsilon \frac{\partial^2 \psi}{\partial x^2} - \epsilon \frac{\partial^2 v_z}{\partial x^2} \right) \quad (30b)$$

and hence,

$$\omega_z \equiv \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\sigma \epsilon (c_1 \nabla^2 \psi + \nabla^2 v_z) \quad (31)$$

where,

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Next, using (29), equation (24c) leads to, to $O(1)$,

$$a = -\nabla^2 \psi. \quad (32)$$

Using (31), and putting

$$\omega_z \equiv \epsilon \sigma c_1 \omega \quad (33)$$

equation (32) leads to

$$a = q \equiv \omega + \frac{1}{c_1} \nabla^2 v_z \quad (34)$$

implying that the torsion coefficient α is proportional to the “*potential vorticity*” q in Hall MHD.

On the other hand, using (23) and (34), equation (27) leads to, to $O(\epsilon)$,

$$\epsilon \sigma c_1 \frac{\partial q}{\partial \xi} + v_x \frac{\partial q}{\partial x} + v_y \frac{\partial q}{\partial y} + \sigma \epsilon [(q - c_1 b), v_z] = 0 \quad (35)$$

where,

$$[f, g] \equiv \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}.$$

If we take the Hall MHD Lagrange multiplier b also to be proportional to the “*potential vorticity*” q , i.e.,

$$b = \frac{1}{c_1} q \quad (36)$$

equation (35) becomes the “*potential vorticity*” conservation equation in 2D hydrodynamics (on identifying ξ with t),

$$\epsilon \sigma c_1 \frac{\partial q}{\partial \xi} + v_x \frac{\partial q}{\partial x} + v_y \frac{\partial q}{\partial y} = 0. \quad (37)$$

Thus, the Beltrami condition (19) in Hall MHD becomes equivalent to the “*potential vorticity*” conservation equation in 2D hydrodynamics if the Hall MHD Lagrange multiplier β is taken to be proportional to the “*potential vorticity*” q as well.³ (34) then implies that the winding pattern of the magnetic field lines in Hall MHD evolves in the same way as “*potential vorticity*” lines in 2D hydrodynamics.

4. Discussion

In this paper, we have extended Parker’s [5] - [7] formulation of isotopological plasma relaxation process in MHD to Hall MHD. The torsion coefficient α in the Hall MHD Beltrami condition turns out now to be proportional to the “*potential vorticity*.” The Hall MHD Beltrami condition becomes equivalent to the “*potential vorticity*” conservation equation in 2D hydrodynamics if the Hall MHD Lagrange multiplier β is taken to be proportional to the “*potential vorticity*” as well. The winding pattern of the magnetic field lines in Hall MHD then appears to evolve in the same way as “*potential vorticity*” lines in 2D hydrodynamics. The analogy between a smooth, continuous magnetic field in Hall MHD and 2D hydrodynamics as in ordinary MHD (Parker [7]) implies that the current sheets seem to have the same role in the development of Hall MHD equilibria as they do in the MHD case.

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³(36) is *sufficient* but not necessary to obtain equation (37).

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